

# FILTRATION LAW FOR POLYMER FLOW THROUGH POROUS MEDIA

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**Abstract.** In this paper we study the filtration laws for the polymeric flow in a porous medium. We use the quasi-Newtonian models with share dependent viscosity obeying the power-law and the Carreau's law. Using the method of homogenization in [5] the coupled micro-macro homogenized law, governing the quasi-newtonian flow in a periodic model of a porous medium, was found. We decouple that law separating the micro from the macro scale. We write the macroscopic filtration law in the form of non-linear Darcy's law and we prove that the obtained law is well posed. We give the analytical as well as the numerical study of our model.

## 1 Introduction

It is well-known that the viscosity of a polymer melt or a polymer solution significantly changes with the share rate (see e.g. [1]). Therefore the Newtonian model, characterized by a constant viscosity, is not a reasonable choice for modelling the polymeric flow. The simplest way to overcome that difficulty is to use the notion of quasi-Newtonian flow, i.e. to modify the Newton's model by using the variable viscosity that depends on the share-rate according to some empirical law. In this paper, in order to describe the polymer flow, we use the share dependent viscosities obeying the power-law

$$\eta(e(u)) = \mu |e(u)|^{r-2} e(u) \quad , \quad 1 < r < 2 \quad , \quad \mu > 0 \quad (1)$$

or the Carreau's law

$$\eta(e(u)) = (\eta_0 - \eta_\infty)(1 + \lambda |e(u)|^2)^{r/2-1} + \eta_\infty \quad , \quad 1 < r < 2 \quad , \quad \eta_0 > \eta_\infty \geq 0 \quad , \quad \lambda > 0 \quad , \quad (2)$$

where

$$e(u) = \frac{1}{2}(\nabla u + \nabla u^t) \quad \text{the rate-of-strain tensor} \quad ,$$

$$|e(u)| = \left( \sum_{i,j=1}^n |e(u)_{ij}|^2 \right)^{1/2} \quad \text{-- the share rate} \quad ,$$

To find the effective law describing the polymer flow through a porous medium  $\Omega_\varepsilon$ <sup>1</sup> Bourgeat and Mikelić in [5] (see also [9] and [13]) used the porous medium with periodic geometry and the homogenization technique called the two-scale convergence. Starting from the microscopic problem

$$-2\operatorname{div}\{\eta(e(u^\varepsilon))e(u^\varepsilon)\} + \nabla p^\varepsilon = \varepsilon^\gamma f \quad \text{in } \Omega_\varepsilon \quad (3)$$

$$\operatorname{div} u^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon \quad (4)$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \quad , \quad (5)$$

depending on  $\gamma$ , the following three types of homogenized problems were rigorously derived:

1. If the Carreau's law was taken for the viscosity and  $\gamma \neq 1$ , the homogenized law is the Darcy's law

$$v(x) = K(f(x) - \nabla_x p^0(x)) \quad , \quad x \in \Omega \quad (6)$$

$$\operatorname{div}_x v = 0 \quad \text{in } \Omega \quad , \quad (7)$$

where  $\Omega$  is the whole domain (the fluid and the solid part),  $v$  is the filtration velocity,  $p^0$  is the pressure and  $K$  is the permeability tensor, depending on the pore structure. In real-world applications  $K$  is a measurable quantity. In our periodic model it can be computed from the cell problem

$$\begin{aligned} -\mu \Delta w^j + \nabla \pi^j &= e_j \quad \text{in } \mathcal{Y} \\ \operatorname{div} w^j &= 0 \quad \text{in } \mathcal{Y} \\ (w^j, \pi^j) &\text{ is } \mathcal{Y}\text{-periodic} \end{aligned} \quad (8)$$

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<sup>1</sup>where  $\varepsilon$  stands for the pore size

by

$$K_{ij} = \int_{\mathcal{Y}} w_j^i = \mu \int_{\mathcal{Y}} \nabla w^i \nabla w^j, \quad (9)$$

where  $\mathcal{Y} \subset ]0, 1[^n$  is the fluid part of the unit cell (the period)  $Y = ]0, 1[^n$  and  $S$  is the boundary of its solid part  $A = Y \setminus \mathcal{Y}$  of  $Y$ . The viscosity  $\mu$  is equal to  $\eta_0$ , in case  $\gamma < 1$  and to  $\eta_\infty$  if  $\gamma > 1$  and  $\eta_\infty > 0$ . Defining

$$u^0(x, y) = \sum_{i=1}^n w^i(y) \frac{\partial p^0}{\partial x_i}(x), \quad p^1(x, y) = \sum_{i=1}^n \pi^i(y) \frac{\partial p^0}{\partial x_i}(x),$$

the above filtration law (6) and the cell problem (8) can be written in the coupled form

$$-\mu \Delta_y u^0 + \nabla_y p^1 = f(x) - \nabla_x p^0(x) \quad \text{in } \Omega \times \mathcal{Y} \quad (10)$$

$$\operatorname{div}_y u^0 = 0 \quad \text{in } \Omega \times \mathcal{Y} \quad (11)$$

$$(u^0, p^1) \text{ is } \mathcal{Y} - \text{periodic in } y \quad (12)$$

$$\operatorname{div}_x \left( \int_{\mathcal{Y}} u^0 dy \right) = 0 \quad \text{in } \Omega \quad (13)$$

$$u^0 = 0 \quad \text{on } \Omega \times S \quad (14)$$

$$\mathbf{n} \cdot \left( \int_{\mathcal{Y}} u^0 dy \right) = 0 \quad \text{on } \partial\Omega. \quad (15)$$

Two functions  $u^0$  and  $p^1$  are, in fact, first oscillating terms in the asymptotic expansion for the velocity and for the pressure, respectively.

2. If the viscosity obeys the Carreau's law with  $\gamma = 1$ , the homogenized law is the coupled Carreau's law

$$-\operatorname{div}_y \{ \eta_c(e_y(u^0)) e_y(u^0) \} + \nabla_y p^1 = f(x) - \nabla_x p^0(x) \quad \text{in } \Omega \times \mathcal{Y} \quad (16)$$

$$\operatorname{div}_y u^0 = 0 \quad \text{in } \Omega \times \mathcal{Y}$$

$$(u^0, p^1) \text{ is } \mathcal{Y} - \text{periodic in } y$$

$$\operatorname{div}_x \left( \int_{\mathcal{Y}} u^0 dy \right) = 0 \quad \text{in } \Omega$$

$$u^0 = 0 \quad \text{on } \Omega \times S$$

$$\mathbf{n} \cdot \left( \int_{\mathcal{Y}} u^0 dy \right) = 0 \quad \text{on } \partial\Omega,$$

$$\eta_c(\xi) = (\eta_0 - \eta_\infty)(1 + \lambda|\xi|^2)^{r/2-1} + \eta_\infty. \quad (17)$$

Here  $p^0$  is the pressure and

$$v = \int_{\mathcal{Y}} u^0(x, y) dy$$

is the filtration velocity. Functions  $u^0(x, y)$  and  $p^1(x, y)$  are oscillating in  $y$  and, in analogy with the linear case, can be seen as the first oscillating terms in the asymptotic expansion of the flow.

3. In case of power-law, the homogenized law is the coupled power-law

$$-\operatorname{div}_y \{ \eta_p(e_y(u^0)) e_y(u^0) \} + \nabla_y p^1 = f(x) - \nabla_x p^0(x) \quad \text{in } \Omega \times \mathcal{Y} \quad (18)$$

$$\operatorname{div}_y u^0 = 0 \quad \text{in } \Omega \times \mathcal{Y}$$

$$(u^0, p^1) \text{ is } \mathcal{Y} - \text{periodic in } y$$

$$\operatorname{div}_x \left( \int_{\mathcal{Y}} u^0 dy \right) = 0 \quad \text{in } \Omega$$

$$\begin{aligned}
u^0 &= 0 \quad \text{on } \Omega \times S \\
\mathbf{n} \cdot \left( \int_{\mathcal{Y}} u^0 dy \right) &= 0 \quad \text{on } \partial\Omega \quad , \\
\eta_p(\xi) &= \mu |\xi|^{r-2} \xi \quad .
\end{aligned} \tag{19}$$

In case 1 two scales are separated and the filtration law is completely macroscopic. All the microscopic information are contained in the permeability tensor  $K$ , which can be measured in applications. In cases 2 and 3 the situation is different and two scales are still coupled. Even if, mathematically speaking, those results solve the homogenization problem for such fluids, to get a physically relevant result one needs to decouple the problems and to find the filtration laws in the usual macroscopic form

$$v = \mathcal{U}(f - \nabla_x p^0) \tag{20}$$

where

$$v(x) = \int_{\mathcal{Y}} u^0(x, y) dy \quad .$$

Using the idea from [3], [4] and [12] to decouple the homogenized problems we define the auxiliary problem

$$-\operatorname{div}_y \{ \eta_\alpha(e_y(w_\xi)) e_y(w_\xi) \} + \nabla_y \pi_\xi = \xi \quad \text{in } \mathcal{Y} \tag{21}$$

$$\operatorname{div}_y w_\xi = 0 \quad \text{in } \mathcal{Y} \tag{22}$$

$$(w_\xi, \pi_\xi) \text{ is } \mathcal{Y} - \text{periodic} \tag{23}$$

$$w_\xi = 0 \quad \text{on } S \quad , \quad \alpha = c, p \quad . \tag{24}$$

With  $(w_\xi, \pi_\xi)$  we define the permeability function  $\mathcal{U} : \mathbf{R}^n \rightarrow \mathbf{R}^n$  by

$$\mathcal{U}(\xi) = \int_{\mathcal{Y}} w_\xi(y) dy \quad , \tag{25}$$

and we obtain formally the filtration law (20). From the definition we can obviously conclude that the function  $\mathcal{U}$  is odd, i.e. that

$$\mathcal{U}(-\xi) = -\mathcal{U}(\xi) \quad .$$

From (18) we also get

$$\operatorname{div} v = 0 \quad \text{in } \Omega \quad , \quad v \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \quad . \tag{26}$$

To show that (18) and (20)-(26) are equivalent, we prove in sections 2 and 4 that  $\mathcal{U}$  is monotone and coercive, consequently, that (20)-(26) is well posed.

We also perform the qualitative analysis of the permeability function  $\mathcal{U}$ . In case of coupled Carreau's law, we find its Taylor's expansion.

## 2 The Carreau's coupled homogenized problem

In this section we use the notation  $\eta$  for the Carreau's viscosity  $\eta_c$ , i.e. we omit the index  $c$ . We begin this section with statement of the main results

## 3 Statement of the main result

To prove that our formal separation of the coupled problem is meaningful we have to prove that our macroscopic problem has a unique solution.

**Theorem 1** *Let  $\mathcal{U} : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be defined by (25). Then the macroscopic problem*

$$\operatorname{div} \mathcal{U}(f - \nabla p^0) = 0 \quad \text{in } \Omega \quad (27)$$

$$\mathbf{n} \cdot \mathcal{U}(f - \nabla p^0) = 0 \quad \text{on } \partial\Omega \quad (28)$$

*has a unique (up to a constant) solution  $p^0 \in W^{1,r'}(\Omega)$ , for any  $f \in L^{r'}(\Omega)^n$ .*

In order to approximate the permeability function  $\mathcal{U}$  in vicinity of 0 we compute its Taylor's expansion. First two terms can be written in the form

$$\mathcal{U}(\xi) = K\xi + \frac{1}{2}(\eta_0 - \eta_\infty)\lambda(2-r) \sum_{m,j,k,\ell=1}^n H_{jk}^{\ell m} \xi_j \xi_k \xi_\ell e_m + O(|\xi|^5), \quad (29)$$

where  $K$  is the classical Darcy's permeability and the coefficients  $H_{jk}^{\ell m}$  are defined by (55). Such approximate filtration law, given by the polynomial permeability

$$\mathcal{V}(\xi) = K\xi + \frac{1}{2}(\eta_0 - \eta_\infty)\lambda(2-r) \sum_{m,j,k,\ell=1}^n H_{jk}^{\ell m} \xi_j \xi_k \xi_\ell e_m$$

is still well posed:

**Theorem 2** *Let  $f \in L^4(\Omega)^n$ . The problem*

$$\operatorname{div} \mathcal{V}(f - \nabla q^0) = 0 \quad \text{in } \Omega \quad (30)$$

$$\mathbf{n} \cdot \mathcal{V}(f - \nabla q^0) = 0 \quad \text{on } \partial\Omega \quad (31)$$

*has a unique (up to a constant) solution  $q^0 \in W^{1,4}(\Omega)$ .*

### 3.1 Taylor's expansion of the permeability function

We want to find the Taylor's expansion for  $\mathcal{U}$  in vicinity of 0. Since the function is odd all the derivatives of a pair order  $D^\alpha \mathcal{U}(0)$ ,  $|\alpha| = 2m$ ,  $m \in \mathbf{N}$  are equal to 0. Deriving (21) with respect to  $\xi_j$  we find

**Theorem 3** *Let  $K$  be the Darcy's permeability tensor computed from the local problem*

$$\begin{aligned} -\eta_0 \Delta w^j + \nabla \pi^j &= e_j \quad \text{in } \mathcal{Y} \\ \operatorname{div} w^j &= 0 \quad \text{in } \mathcal{Y} \\ (w^j, \pi^j) &\text{ is } \mathcal{Y} - \text{periodic} \end{aligned} \quad (32)$$

by

$$K_{ij} = \int_{\mathcal{Y}} w_j^i = \eta_0 \int_{\mathcal{Y}} \nabla w^i \nabla w^j. \quad (33)$$

Then

$$\nabla_\xi \mathcal{U}(0) = K \quad (34)$$

To prove that we proceed as in [3], [4] or [12]. We begin by recalling the result from [14]:

**Lemma 1** *There exist a constant  $c_1^r > 0$  such that :*

$$c_1^r \frac{|e(v-u)|_{L^r(\mathcal{Y})}^2}{1 + |e(u)|_{L^r(\mathcal{Y})}^{2-r} + |e(v)|_{L^r(\mathcal{Y})}^{2-r}} \leq \int_{\mathcal{Y}} [\eta(e(u))e(u) - \eta(e(v))e(v)] e(u-v) \quad (35)$$

for any

$$u, v \in W = \{\phi \in W^{1,r}(\mathcal{Y})^n ; \operatorname{div} \phi = 0, \phi \text{ is } \mathcal{Y} - \text{periodic}, \phi = 0 \text{ on } S\}.$$

**Lemma 2** Let  $(w_\xi, \pi_\xi) \in W \times L^2(\mathcal{Y})$  be defined by (21)-(24). Then

$$|e(w_\xi)|_{L^r(\mathcal{Y})} \leq 2 \frac{C^r}{c_1^r} |\mathcal{Y}|^{1-\frac{1}{r}} |\xi|, \quad (36)$$

for any  $\xi$  such that

$$|\xi| < \frac{c_1^r}{2C^r |\mathcal{Y}|^{1-\frac{1}{r}}},$$

where  $C^r$  is the Poincaré-Korn's constant in  $W$  (i.e. such that  $|\phi|_{L^r(\mathcal{Y})} \leq C^r |e(\phi)|_{L^r(\mathcal{Y})} \forall \phi \in W$ ).

**Proof.** Multiplying (21) by  $w_\xi$  and integrating over  $\mathcal{Y}$  we obtain

$$\xi \cdot \int_{\mathcal{Y}} w_\xi = \int_{\mathcal{Y}} \eta(e(w_\xi)) e(w_\xi) \geq c_1^r \frac{|e(w_\xi)|_{L^r(\mathcal{Y})}^2}{1 + |e(w_\xi)|_{L^r(\mathcal{Y})}^{2-r}}.$$

Now (35) gives

$$c_1^r |e(w_\xi)|_{L^r(\mathcal{Y})} \leq C^r |\mathcal{Y}|^{1-\frac{1}{r}} (1 + |e(w_\xi)|_{L^r(\mathcal{Y})}^{2-r}) |\xi|.$$

If  $|e(w_\xi)|_{L^r(\mathcal{Y})} < 1$  then (36) obviously holds. On the other hand, supposing that  $|e(w_\xi)|_{L^r(\mathcal{Y})} > 1$ , implies

$$c_1^r |e(w_\xi)|_{L^r(\mathcal{Y})} \leq 2C^r |\mathcal{Y}|^{1-\frac{1}{r}} |e(w_\xi)|_{L^r(\mathcal{Y})}^{2-r} |\xi|$$

and

$$|e(w_\xi)|_{L^r(\mathcal{Y})}^{r-1} \leq 2 \frac{C^r}{c_1^r} |\mathcal{Y}|^{1-\frac{1}{r}} |\xi|$$

which contradicts the assumption that

$$|\xi| < \frac{c_1^r}{2C^r |\mathcal{Y}|^{1-\frac{1}{r}}} \cdot \clubsuit$$

**Proof of theorem 3.** To prove the claim we only need to show that, as  $|h| \rightarrow 0$

$$|w_h - W(h)|_W = o(|h|), \quad W(h) = \sum_{i=1}^n h_i w^i \quad (37)$$

because

$$|\mathcal{U}(h) - K h| = \left| \int_{\mathcal{Y}} [w_h - W(h)] \right| \leq C |w_h - W(h)|_W. \quad (38)$$

In fact we shall prove that

$$|\mathcal{U}(h) - K h| = O(|h|^3) \quad (39)$$

which implies, not only that  $\nabla \mathcal{U}(0) = K$ , but also that  $\nabla^2 \mathcal{U}(0) = 0$  (which we knew before because  $\mathcal{U}$  is an odd function). Using the standard regularity result (see e.g. [6]) for the problem (32) we see that

$$|W(h)|_{H^3(\mathcal{Y})} \leq \sum_{i=1}^n |h| |w^i|_{H^3(\mathcal{Y})} \leq C |h|. \quad (40)$$

and therefore

$$|W(h)|_{C^1(\overline{\mathcal{Y}})} \leq C |h|. \quad (41)$$

Functions  $W(h)$  and  $\Pi(h) = \sum_{i=1}^n h_i \pi^i$  satisfy the system

$$-\operatorname{div}\{\eta\{e[W(h)]\}e[W(h)]\} + \nabla \Pi(h) = h + R_h \quad (42)$$

$$\operatorname{div} W(h) = 0 \quad (43)$$

$$(W(h), \Pi(h)) \text{ is } \mathcal{Y} \text{-periodic, } W(h) = 0 \text{ on } S, \quad (44)$$

where

$$R_h = \operatorname{div}\{(\eta_0 - \eta_\infty)[(1 + \lambda|e[W(h)]|^2)^{r/2-1} - 1]e[W(h)]\} .$$

An easy computation leads to

$$(1 + \lambda t^2)^{r/2-1} - 1 = (r/2 - 1)t^2 \int_0^\lambda (1 + \tau t^2)^{r/2-2} d\tau \leq C t^2 .$$

For  $1 < r < 2$  we get using (41)

$$\int_{\mathcal{Y}} R_h \phi \leq C |\phi|_{W^{1,r}(\mathcal{Y})} |e[W(h)]|_{L^{3r'}(\mathcal{Y})}^3 \leq C |h|^3 |\phi|_{W^{1,r}(\mathcal{Y})} . \quad (45)$$

Subtracting (21) from (42), multiplying by  $W(h) - w_h$  and integrating over  $\mathcal{Y}$  we obtain

$$\begin{aligned} J &= \int_{\mathcal{Y}} \{\eta\{e[W(h)]\}e[W(h)] - \eta\{e(w_h)\}e(w_h)\} e(W(h) - w_h) = \\ &= \int_{\mathcal{Y}} R_h(W(h) - w_h) \leq C |h|^3 |W(h) - w_h|_W . \end{aligned}$$

An application of lemma 1 gives

$$J \geq c_1 \frac{|W(h) - w_h|_W^2}{1 + C|h|} . \clubsuit$$

As we have seen the second derivative in  $\xi = 0$  is 0. Third can be computed from the auxiliary problem

$$-\eta_0 \Delta w^{\ell j k} + \nabla \pi^{\ell j k} = \quad (46)$$

$$= (\eta_0 - \eta_\infty)(r - 2)\lambda \operatorname{div}\{e(w^k) \cdot e(w^j) e(w^\ell) + \quad (47)$$

$$+ e(w^k) \cdot e(w^\ell) e(w^j) + e(w^\ell) \cdot e(w^j) e(w^k)\} \text{ in } \mathcal{Y} \quad (48)$$

$$\operatorname{div} w^{ijk} = 0 \text{ in } \mathcal{Y} \quad (49)$$

$$(w^{ijk}, \pi^{ijk}) \text{ is } \mathcal{Y} - \text{periodic} \quad (50)$$

by

$$\frac{\partial^3 \mathcal{U}(0)}{\partial \xi_\ell \partial \xi_j \partial \xi_k} = \int_{\mathcal{Y}} w^{\ell j k} .$$

Since

$$\frac{\partial^3 \mathcal{U}(0)_m}{\partial \xi_\ell \partial \xi_j \partial \xi_k} = \eta_0 \int_{\mathcal{Y}} \nabla w^{\ell j k} \nabla w^m = \quad (51)$$

$$= (\eta_0 - \eta_\infty)(2 - r)\lambda \int_{\mathcal{Y}} \{e(w^k) \cdot e(w^j) e(w^\ell) \cdot e(w^m) + \quad (52)$$

$$+ e(w^k) \cdot e(w^\ell) e(w^j) \cdot e(w^m) + e(w^\ell) \cdot e(w^j) e(w^k) \cdot e(w^m)\} = \quad (53)$$

$$= (\eta_0 - \eta_\infty)(2 - r)\lambda (H_{jk}^{\ell m} + H_{\ell k}^{jm} + H_{\ell j}^{km}) , \quad (54)$$

where

$$H_{jk}^{\ell m} = H_{jk}^{m\ell} = H_{kj}^{\ell m} = H_{\ell m}^{jk} = \int_{\mathcal{Y}} e(w^k) \cdot e(w^j) e(w^\ell) \cdot e(w^m) . \quad (55)$$

This gives the expansion for  $\mathcal{U}$  in the form

$$\mathcal{U}(\xi) = K\xi + \frac{1}{2}(\eta_0 - \eta_\infty)\lambda(2 - r) \sum_{m,j,k,\ell=1}^n H_{jk}^{\ell m} \xi_j \xi_k \xi_\ell e_m + O(|\xi|^5) . \quad (56)$$

**Proposition 1** *Let  $H_{jk}^{\ell m}$  be defined by (55). Then (56) holds.*

**Proof.** We use the same idea as in the proof of theorem 3. We define

$$\begin{aligned} V(h) &= \sum_{i=1}^n w^i h_i + \frac{1}{2} \sum_{i,j,k=1}^n w^{ijk} h_i h_j h_k \\ Q(h) &= \sum_{i=1}^n \pi^i h_i + \frac{1}{2} \sum_{i,j,k=1}^n \pi^{ijk} h_i h_j h_k . \end{aligned}$$

Now

$$\begin{aligned} & -\operatorname{div}\{\eta_r(e[V(h)])e(V(h))\} + \nabla Q(h) = \\ & = h - (\eta_0 - \eta_\infty)\operatorname{div}\{[1 + \lambda|e(V(h))|^2]^{r/2-1} - 1\}e[V(h)] + \\ & + \frac{\lambda}{2}(r-2)(\eta_0 - \eta_\infty)\operatorname{div}\left\{\sum_{i,j,m=1}^n e(w^i) \cdot e(w^j) e(w^m) h_i h_j h_m\right\} . \end{aligned}$$

It only remains to estimate

$$\begin{aligned} J &= (\eta_0 - \eta_\infty)\operatorname{div}\{[1 + \lambda|e(V(h))|^2]^{r/2-1} - 1\}e[V(h)] + \\ & + \frac{\lambda}{2}(r-2)(\eta_0 - \eta_\infty)\operatorname{div}\left\{\sum_{i,j,m=1}^n e(w^i) \cdot e(w^j) e(w^m) h_i h_j h_m\right\} . \end{aligned}$$

A simple Taylor's formula gives

$$(1 + \lambda|\xi|^2)^{r/2-1} - 1 = \frac{1}{2}\lambda|\xi|^2 + O(\lambda^2|\xi|^4) .$$

After recalling that  $e(w^i) \in C(\overline{\mathcal{Y}})^{n \times n}$ ,  $e(w^{ijk}) \in C(\overline{\mathcal{Y}})^{n \times n}$ , we get for any  $y \in \mathcal{Y}$

$$\begin{aligned} & (\eta_0 - \eta_\infty)\operatorname{div}\{[1 + \lambda|e(V(h))|^2]^{r/2-1} - 1\}e[V(h)] = \\ & = \frac{\lambda}{2}(r-2)(\eta_0 - \eta_\infty)\operatorname{div}\left\{\sum_{i,j,m=1}^n e(w^i) \cdot e(w^j) e(w^m) h_i h_j h_m\right\} + O(|h|^4) . \end{aligned}$$

Arguing as in the proof of theorem 3, we get the claim. ♣

**Remark 1** *It can be seen from the definition that  $H_{jj}^{ii} > 0$ ,  $H_{\ell k}^{\ell k} > 0$ . Furthermore, for*

$$Z(h) = \sum_{i,j,k,\ell=1}^n H_{jk}^{i\ell} h_j h_k h_i \mathbf{e}_\ell = \int_{\mathcal{Y}} \left| \sum_{k=1}^n e(w^k) h_k \right|^2 \sum_{i=1}^n e(w^i) h_i \sum_{\ell=1}^n e(w^\ell) \mathbf{e}_\ell$$

we have

$$Z(h) \cdot h = \int_{\mathcal{Y}} \left| \sum_{k=1}^n e(w^k) h_k \right|^2 > 0 .$$

To compute further order terms we proceed in the same way and we get another auxiliary problem

$$-\eta_0 \Delta w^{ijk\ell p} + \nabla \pi^{ijk\ell p} = \tag{57}$$

$$= (\eta_0 - \eta_\infty) \lambda (r-2) \operatorname{div}\{ \tag{58}$$

$$e(w^i) \cdot e(w^j) e(w^{k\ell m}) + e(w^k) \cdot e(w^i) e(w^{j\ell m}) + e(w^i) \cdot e(w^\ell) e(w^{jk m}) + \tag{59}$$

$$e(w^j) \cdot e(w^p) e(w^{jk\ell}) + e(w^j) \cdot e(w^k) e(w^{i\ell m}) + e(w^j) \cdot e(w^\ell) e(w^{ik m}) + \tag{60}$$

$$e(w^j) \cdot e(w^p) e(w^{ik\ell}) + e(w^k) \cdot e(w^\ell) e(w^{ij m}) + e(w^k) \cdot e(w^p) e(w^{ij\ell}) + \tag{61}$$



$$e(w^\ell) \cdot e(w^p) e(w^{ijk}) + \quad (62)$$

$$(e(w^{jkl}) \cdot e(w^i) + e(w^{ikl}) \cdot e(w^j) + e(w^{ijl}) \cdot e(w^k) + e(w^{ijk}) \cdot e(w^\ell)) e(w^p) + \quad (63)$$

$$(e(w^{jkm}) \cdot e(w^i) + e(w^{ikm}) \cdot e(w^j) + e(w^{ijm}) \cdot e(w^k) + e(w^{ijk}) \cdot e(w^p)) e(w^\ell) + \quad (64)$$

$$(e(w^{jlm}) \cdot e(w^i) + e(w^{ilm}) \cdot e(w^j) + e(w^{ijm}) \cdot e(w^\ell) + e(w^{ijl}) \cdot e(w^p)) e(w^k) + \quad (65)$$

$$(e(w^{klm}) \cdot e(w^i) + e(w^{ilm}) \cdot e(w^k) + e(w^{ikm}) \cdot e(w^\ell) + e(w^{ikl}) \cdot e(w^p)) e(w^j) + \quad (66)$$

$$(e(w^{klm}) \cdot e(w^j) + e(w^{jlm}) \cdot e(w^k) + e(w^{jkm}) \cdot e(w^\ell) + e(w^{jkl}) \cdot e(w^p)) e(w^i) + \quad (67)$$

$$\lambda(r-4) \{ \quad (68)$$

$$\{e(w^\ell) \cdot e(w^k) e(w^j) \cdot e(w^i) + e(w^\ell) \cdot e(w^j) e(w^k) \cdot e(w^i) + e(w^\ell) \cdot e(w^i) e(w^k) \cdot e(w^j)\} e(w^p) + \quad (69)$$

$$\{e(w^p) \cdot e(w^k) e(w^j) \cdot e(w^i) + e(w^p) \cdot e(w^j) e(w^k) \cdot e(w^i) + e(w^p) \cdot e(w^i) e(w^k) \cdot e(w^j)\} e(w^\ell) + \quad (70)$$

$$\{e(w^p) \cdot e(w^\ell) e(w^j) \cdot e(w^i) + e(w^p) \cdot e(w^j) e(w^\ell) \cdot e(w^i) + e(w^p) \cdot e(w^i) e(w^\ell) \cdot e(w^j)\} e(w^k) + \quad (71)$$

$$\{e(w^p) \cdot e(w^\ell) e(w^k) \cdot e(w^i) + e(w^p) \cdot e(w^k) e(w^\ell) \cdot e(w^i) + e(w^p) \cdot e(w^i) e(w^\ell) \cdot e(w^k)\} e(w^j) + \quad (72)$$

$$\{e(w^p) \cdot e(w^\ell) e(w^k) \cdot e(w^j) + e(w^p) \cdot e(w^k) e(w^\ell) \cdot e(w^j) + e(w^p) \cdot e(w^j) e(w^\ell) \cdot e(w^k)\} e(w^i) \} \quad (73)$$

$$\operatorname{div} w^{ijk\ell p} = 0 \text{ in } \mathcal{Y} \quad (74)$$

$$(w^{ijk\ell p}, \pi^{ijk\ell p}) \text{ is } \mathcal{Y} - \text{periodic} . \quad (75)$$

Now

$$\frac{\partial^5 \mathcal{U}(0)}{\partial \xi_i \partial \xi_j \partial \xi_k \partial \xi_\ell \partial \xi_p} = \int_{\mathcal{Y}} w^{ijk\ell p}$$

and we get an expansion in the form

$$\begin{aligned} \mathcal{U}(\xi) &= K\xi + \frac{1}{2}(\eta_0 - \eta_\infty)\lambda(2-r) \sum_{i,j,k,\ell,m=1}^n H_{jk}^{\ell m} \xi_j \xi_k \xi_\ell \mathbf{e}_m + \\ &+ \frac{1}{2}(\eta_0 - \eta_\infty)\lambda(2-r) \sum_{j,k,\ell,m,p,q=1}^n H_{jkl}^{mpq} \xi_j \xi_k \xi_\ell \xi_m \xi_p \mathbf{e}_q + O(|\xi|^7) \end{aligned}$$

where

$$H_{jkl}^{mpq} = \int_{\mathcal{Y}} \{e(w^j) \cdot e(w^k) e(w^{\ell mp}) \cdot e(w^q) + \lambda \frac{(r-4)}{2} e(w^j) \cdot e(w^k) e(w^\ell) \cdot e(w^m) e(w^p) \cdot e(w^q)\} .$$

This computation can be proceeded.

### 3.2 Proofs of existence theorems

To prove the theorem 3 we first prove the following lemma that gives coercivity and boundness of the differential operator

$$A(\phi) = -\operatorname{div} \mathcal{U}(f - \nabla \phi) \ .$$

**Lemma 3** *There exist constants  $M, C, c > 0$  such that*

$$\mathcal{U}(\xi) \cdot \xi \geq c|\xi|^{r'} \text{ for any } |\xi| > M \quad (76)$$

$$|\mathcal{U}(\xi)| \leq C|\xi|^{r'-1} \text{ for any } \xi \in \mathbf{R}^n \ . \quad (77)$$

**Proof.** The upper bound is easy to prove, since we have, due to (35)

$$c_1^r \frac{|e(w_\xi)|_{L^r(\mathcal{Y})}}{|e(w_\xi)|_{L^r(\mathcal{Y})}^{2-r} + 1} \leq C|\xi| \ .$$

To prove the lower bound (76) we define  $v_\xi = |\xi|^{1-r'} w_\xi$  and  $q_\xi = |\xi|^{-1} \xi$ . Those functions satisfy the system

$$-\operatorname{div}_y \{ \eta_\xi(e_y(v_\xi))e_y(v_\xi) \} + \nabla_y q_\xi = \frac{\xi}{|\xi|} \text{ in } \mathcal{Y} \quad (78)$$

$$\operatorname{div}_y v_\xi = 0 \text{ in } \mathcal{Y} \quad (79)$$

$$(v_\xi, q_\xi) \text{ is } \mathcal{Y} - \text{periodic} \quad (80)$$

$$v_\xi = 0 \text{ on } S \ , \quad (81)$$

where

$$\eta_\xi(\tau) = (\eta_0 - \eta_\infty)(|\xi|^{2(1-r')} + \lambda|\tau|^2)^{r/2-1} + \eta_\infty|\xi|^{-r'} \ .$$

Applying again (35) we obtain

$$\frac{|e(v_\xi)|_{L^r(\mathcal{Y})}}{|\xi|^{2(1-r')(2-r)} + |e(v_\xi)|_{L^r(\mathcal{Y})}} \leq C$$

so that we can extract a subsequence  $\{\xi_n\}$  such that  $|\xi_n| \rightarrow \infty$ ,  $|\xi_n|^{-1} \xi_n \rightarrow \xi_\infty$ ,  $|\xi_\infty| = 1$  and

$$v_{\xi_n} \rightharpoonup v_\infty \text{ weakly in } W \text{ as } n \rightarrow \infty \ .$$

Using the Minty's lemma we get that  $v_\infty$  satisfies the system

$$-(\eta_0 - \eta_\infty)\lambda^{r/2-1} \operatorname{div}_y \{ |e_y(v_\infty)|^{r-2} e_y(v_\infty) \} + \nabla_y q_\infty = \xi_\infty \text{ in } \mathcal{Y}$$

$$\operatorname{div}_y v_\infty = 0 \text{ in } \mathcal{Y}$$

$$(v_\infty, q_\infty) \text{ is } \mathcal{Y} - \text{periodic}$$

$$v_\infty = 0 \text{ on } S \ ,$$

For any  $\xi_\infty$ , such that  $|\xi_\infty| = 1$ , we can define a continuous function

$$\mathcal{I}(\xi_\infty) = \int_{\mathcal{Y}} v_\infty \neq 0 \ .$$

Furthermore

$$\mathcal{I}(\xi_\infty) \cdot \xi_\infty = \int_{\mathcal{Y}} (\eta_0 - \eta_\infty) \lambda^{r/2-1} |e(v_\infty)|^r \ .$$

and there exists

$$c_\infty = \inf_{|\tau|=1} \mathcal{I}(\tau) \cdot \tau > 0 \ .$$

Two functionals

$$\begin{aligned}\Phi_\xi(v) &= \int_{\mathcal{Y}} \eta_\xi(e(v)) |e(v)|^2 \\ \Phi_\infty(v) &= \int_{\mathcal{Y}} (\eta_0 - \eta_\infty) \lambda^{r/2-1} |e(v)|^r\end{aligned}$$

are obviously convex and, due to the Lebesgues dominated convergence theorem,

$$\Phi_\xi(v_\xi) - \Phi_\infty(v_\xi) \rightarrow 0 \quad \text{as } |\xi| \rightarrow \infty .$$

Since  $\Phi_\infty$  is weak lower semicontinuous in  $W$ , we obtain

$$\lim_{|\xi| \rightarrow \infty} \inf \Phi_\xi(v_\xi) = \lim_{|\xi| \rightarrow \infty} \inf \{\Phi_\infty(v_\xi) + [\Phi_\xi(v_\xi) - \Phi_\infty(v_\xi)]\} \geq \Phi_\infty(v_\infty) \geq c_\infty > 0 .$$

But

$$\Phi_\xi(v_\xi) = |\xi|^{-r'} \mathcal{U}(\xi) \cdot \xi$$

which proves the claim. ♣

**Proof of theorem 3.** Lemma 3.2 implies coercivity and boundness of the operator  $A : W^{1,r'}(\Omega)/\mathbf{R} \rightarrow (W^{1,r'}(\Omega))'$ . Its strict monotonicity follows from inequality

$$\begin{aligned}[\mathcal{U}(\xi) - \mathcal{U}(\tau)] \cdot (\xi - \tau) &= \int_{\mathcal{Y}} \{\eta_c(e(w_\xi))e(w_\xi) - \eta_c(w_\tau)\} \cdot (w_\xi - w_\tau) \\ &\geq c_1^r \frac{|e(w_\xi - w_\tau)|_{L^r(\mathcal{Y})}^2}{|e(w_\xi)|_{L^r(\mathcal{Y})}^{2-r} + |e(w_\tau)|_{L^r(\mathcal{Y})}^{2-r} + 1} .\end{aligned}$$

Now the classical Browder-Minty's theorem proves the existence of the solution, i.e. the surjectivity of our operator. As it is strictly monotone, it is also injective, i.e. the solution is unique. ♣

If we approximate  $\mathcal{U}$  by taking only first two terms in its Taylor's series, i.e., if we consider

$$\mathcal{V}(\xi) = K\xi + \frac{1}{2}(\eta_0 - \eta_\infty)\lambda(2-r) \sum_{m,j,k,\ell=1}^n H_{jk}^{\ell m} \xi_j \xi_k \xi_\ell e_m$$

instead of  $\mathcal{U}$ , we get the problem (30)-(31).

For such approximation of the filtration law we have theorem 2, from the beginning of this section. We are now able to prove it.

**Proof of theorem 2.** We define the formal differential operator

$$B(\varphi) = \text{div } \mathcal{V}(f - \nabla \varphi) .$$

Since  $\mathcal{V}$  is a polynomial of order 3, it satisfies the estimate

$$|\mathcal{V}(\xi)| \leq C (|\xi| + |\xi|^3) .$$

Consequently  $B : W^{1,4}(\Omega)/\mathbf{R} \rightarrow (W^{1,4}(\Omega)/\mathbf{R})'$  is a bounded operator. Moreover

$$|B(\varphi)|_{(W^{1,4}(\Omega)/\mathbf{R})'} \leq C |f - \nabla \varphi|_{L^4(\Omega)}^3 .$$

Next we prove the coercivity. Let

$$E(\xi) = \sum_{i=1}^n e(w^i) \xi_i .$$

Then, for any  $\xi \in \mathbf{R}^n$ ,

$$\mathcal{V}(\xi) \cdot \xi = \int_{\mathcal{Y}} ( \eta_0 |E(\xi)|^2 + (\eta_0 - \eta_\infty) \lambda(1 - r/2) |E(\xi)|^4 ) .$$

The functions

$$\mathcal{G}(\xi) = \int_{\mathcal{Y}} |E(\xi)|^2 \quad , \quad \mathcal{H}(\xi) = \int_{\mathcal{Y}} |E(\xi)|^4$$

are continuous and strictly positive for  $\xi \neq 0$ . Therefore there exist

$$\kappa_1 = \inf_{|\xi|=1} \mathcal{G}(\xi) \quad , \quad \kappa_2 = \inf_{|\xi|=1} \mathcal{H}(\xi) \quad .$$

But than

$$\mathcal{V}(\xi) \cdot \xi \geq \kappa_1 |\xi|^2 + (\eta_0 - \eta_\infty) \lambda (1 - r/2) \kappa |\xi|^4 \quad ,$$

implying the coercivity of  $B$  on  $W^{1,4}(\Omega)/\mathbf{R}$ . It remains to prove the monotonicity of  $B$ . Then the result will follow from the classical Browder-Minty's theorem.

For any  $\xi, \tau \in \mathbf{R}^n$  the derivative  $D\mathcal{V}(\xi)$  satisfies

$$\begin{aligned} D\mathcal{V}(\xi) \tau \cdot \tau &= K\tau \cdot \tau + \frac{1}{2}\lambda(2-r)(\eta_0 - \eta_\infty) \int_{\mathcal{Y}} [(E(\xi) \cdot E(\tau))^2 + \\ &+ E(\xi)^2 E(\tau)^2] > K\tau \cdot \tau \geq k_0 |\tau|^2 \quad . \end{aligned}$$

The above inequality proves that the equation (30) is uniformly elliptic , i.e. that the operator  $B$  is monotone. ♣

The approximation  $\mathcal{V}$  of the permeability function is valid either for small  $|\xi|$  or for small  $\lambda$ . For large  $\lambda$  and  $|\xi|$ , the Carreau's auxiliary problem (21)-(24), with  $\eta_c$ , is close to the power-law auxiliary problem (21)-(24), with  $\eta_p$  and  $\mu = (\eta_0 - \eta_\infty)\lambda^{r/2-1}$ . We study the power-law auxiliary problem in the next section.

We refer to section 6.3 for numerical computation of the permeability function  $\mathcal{U}$  and its approximation  $\mathcal{V}$  and their comparison for sufficiently small  $|\xi|$ .

## 4 The power-law coupled homogenized problem

In this section we also start with statement of its main results and we adopt the notation  $\eta$  for the power-law viscosity  $\eta_p$ .

## 5 Statement of the main result

From (21)-(25) we get by substitution that the permeability function is homogeneous in the sense that

$$\mathcal{U}(\lambda \xi) = |\lambda|^{r'-2} \lambda \mathcal{U}(\xi) \quad , \quad \lambda \in \mathbf{R} \quad . \quad (82)$$

implying

$$m|\xi|^{r'-1} \leq |\mathcal{U}(\xi)| \leq M|\xi|^{r'-1} \quad (83)$$

with

$$m = \inf_{|\xi|=1} |\mathcal{U}(\xi)| \quad , \quad M = \sup_{|\xi|=1} |\mathcal{U}(\xi)| \quad .$$

Now, as in the previous chapter, (83) and (88) imply that:

**Theorem 4** *For any  $f \in L^{r'}(\Omega)^n$  the macroscopic problem*

$$\operatorname{div} \mathcal{U}(f - \nabla p^0) = 0 \quad \text{in } \Omega \quad (84)$$

$$\mathbf{n} \cdot \mathcal{U}(f - \nabla p^0) = 0 \quad \text{on } \partial\Omega \quad (85)$$

*has a unique (up to a constant) solution  $p^0 \in W^{1,r'}(\Omega)$ .*

The standard engineering model (see e.g. [2], [15]) suggests that the permeability function  $\mathcal{U}$  can be written in the form

$$\mathcal{U}(\xi)_i = |(K \xi)_i|^{r'-2} (K \xi)_i, \quad i = 1, \dots, n, \quad (86)$$

where  $K$  is a permeability tensor.

To check that conjecture we define the conjugated function  $G$  by

$$G(\xi)_i = |\mathcal{U}(\xi)_i|^{r-2} \mathcal{U}(\xi)_i,$$

such that

$$\mathcal{U}(\xi)_i = |G(\xi)_i|^{r'-2} G(\xi)_i.$$

Now, if the conjecture (86) is true, the function  $G$  is linear. Homogeneity (82) of  $\mathcal{U}$  implies that  $G$  is homogeneous with order 1, i.e. that

$$G(\lambda \xi) = \lambda G(\xi), \quad \lambda \in \mathbf{R}.$$

*Function homogeneous with order 1 is linear if and only if it is differentiable for  $\xi = 0$ .*

Function  $G$  is differentiable for any  $\xi \neq 0$ , but, in general, not for  $\xi = 0$ . That means that **the conjecture (86) is, in general, false**. In case of unidirectional flow it is differentiable and, therefore, linear. In case of a porous medium consisting of a net of narrow channels,  $G$  allows an asymptotic expansion in powers of the thickness of the channels, with the first term being a linear function. Numerical examples show that  $G$  is linear (or differentiable for  $\xi = 0$ ) in case of periodic porous medium with the solid part being an array of rectangular obstacles (i.e. with the fluid part being a net of perpendicular channels). On the other hand, in case of spherical or elliptical obstacle, the function  $G$  is not linear and conjecture (86) does not hold.

## 5.1 Study of the permeability function

We next recall the result from [14], that will be used in the sequel:

**Lemma 4** *There exist a constant  $c_1^r > 0$  such that :*

$$c_1^r \frac{|e(v-u)|_{L^r(\mathcal{Y})}^2}{|e(u)|_{L^r(\mathcal{Y})}^{2-r} + |e(v)|_{L^r(\mathcal{Y})}^{2-r}} \leq \int_{\mathcal{Y}} [\eta(e(u))e(u) - \eta(e(v))e(v)] e(u-v) \quad (87)$$

for any

$$u, v \in W = \{\phi \in W^{1,r}(\mathcal{Y})^n; \operatorname{div} \phi = 0, \phi \text{ is } \mathcal{Y}\text{-periodic}, \phi = 0 \text{ on } S\}.$$

It implies that, for  $\xi, \tau \in \mathbf{R}^n$ ,  $\xi \neq \tau$ ,

$$[\mathcal{U}(\xi) - \mathcal{U}(\tau)] \cdot (\xi - \tau) \geq c_1^r \frac{|e(w_\xi - w_\tau)|_{L^r(\mathcal{Y})}^2}{|e(w_\xi)|_{L^r(\mathcal{Y})}^{2-r} + |e(w_\tau)|_{L^r(\mathcal{Y})}^{2-r}} > 0. \quad (88)$$

Before we continue we give two simple examples mentioned in the introduction of this section.

**Example 1** *As it was noticed in [5], [13], [9], if the flow is unidirectional, e.g. in the direction  $\mathbf{e}_1$ , then  $\mathcal{U}(\xi) = \mathcal{J}(\xi_1) \mathbf{e}_1$ , with*

$$\mathcal{J}(\xi_1) = |\xi_1|^{r'-2} \xi_1.$$

*This result corresponds to the usual models that can be found in the engineering literature (see e.g. [15], [7]).*

**Example 2** In case of a porous medium consisting of a network of thin pipes we can find the asymptotic form of our function  $\mathcal{U}$  and, again, it corresponds to the usual engineering model from [7] and [15]. Suppose that  $n = 2$  and that  $\mathcal{Y} = \mathcal{Y}_\delta$  consists of two narrow perpendicular channels

$$P_\delta^1 = ]0, 1[ \times ] - \delta/2, \delta/2[ \quad , \quad P_\delta^2 = ] - \delta/2, \delta/2[ \times ]0, 1[ \quad ,$$

with thickness  $\delta$ . We consider our auxiliary problem in such unit cell  $\mathcal{Y}_\delta = P_\delta^1 \cup P_\delta^2$

$$-div_y \{ \eta_p(e_y(w_\xi^\delta)) e_y(w_\xi^\delta) \} + \nabla_y \pi_\xi^\delta = \xi \quad \text{in } \mathcal{Y}_\delta \quad (89)$$

$$div_y w_\xi^\delta = 0 \quad \text{in } \mathcal{Y}_\delta \quad (90)$$

$$(w_\xi, \pi_\xi) \text{ is } \mathcal{Y}_\delta - \text{periodic} \quad (91)$$

$$w_\xi = 0 \quad \text{on } S^\delta \quad , \quad (92)$$

where  $S^\delta = \partial(P_\delta^1 \cup P_\delta^2)$ . We study the limit as  $\delta \rightarrow 0$ . The similar problem was considered in [11] for the Newtonian fluid and an explicit formula for the permeability was found. We are going to do the same thing with our nonlinear permeability function  $\mathcal{U}^\delta(\xi) = \int_{\mathcal{Y}} w_\xi^\delta$ . As in [11], we seek an approximation of  $(w_\xi^\delta, \pi_\xi^\delta)$  in the form

$$w(\delta) = \delta^{r'} \left\{ \begin{array}{l} \theta(y_2/\delta) |\xi_1|^{r'-2} \xi_1 \mathbf{e}_1 \text{ in } P_1^\delta \\ \theta(y_1/\delta) |\xi_2|^{r'-2} \xi_2 \mathbf{e}_2 \text{ in } P_2^\delta \end{array} \right. , \quad \pi(\delta) = \left\{ \begin{array}{l} y_2 \xi_2 \text{ in } P_1^\delta \\ y_1 \xi_1 \text{ in } P_2^\delta \end{array} \right.$$

where

$$\theta(z) = \frac{2}{r' \mu} \left( \frac{\sqrt{2}}{\mu} \right)^{r'-2} [ (1/2)^{r'} - |z|^{r'} ] \quad .$$

In each  $P_\delta^i$  we have the Poincaré-Korn's inequality

$$|w_\xi^\delta - w(\delta)|_{L^r(P_\delta^i)} \leq C \delta |e(w_\xi^\delta - w(\delta))|_{L^r(P_\delta^i)}$$

and the trace inequality

$$|w_\xi^\delta - w(\delta)|_{L^r(\Sigma_\delta^i)} \leq C \delta^{1/r'} |e(w_\xi^\delta - w(\delta))|_{L^r(P_\delta^i)} \quad ,$$

where

$$\Sigma_\delta^1 = \partial P_\delta^1 \cap P_\delta^2 \quad , \quad \Sigma_\delta^2 = \partial P_\delta^2 \cap P_\delta^1 \quad .$$

By a direct computation we obtain

$$\begin{aligned} & \mu \int_{P_\delta^i} \{ |e(w_\xi^\delta)|^{r-2} e(w_\xi^\delta) - |e(w(\delta))|^{r-2} e(w(\delta)) \} e(w_\xi^\delta - w(\delta)) = \\ & = \int_{\Sigma_\delta^i} \{ \pi(\delta) \mathbf{n} - \mu |e(w(\delta))|^{r-2} e(w(\delta)) \mathbf{n} \} (w_\xi^\delta - w(\delta)) \quad . \end{aligned}$$

We use (87) to estimate from below the left-hand side and the above inequalities to estimate from above the right-hand side. It leads to the estimate

$$|\delta^{-(1+r')} \mathcal{U}_\delta(\xi) - \langle \theta \rangle \mathcal{J}(\xi)| \leq C \delta^{1/r} \quad ,$$

where

$$\langle \theta \rangle = \int_{-1/2}^{1/2} \theta(s) ds = \frac{2}{(r' + 1)\mu} (\sqrt{2}/\mu)^{r'-2} \quad ,$$

and

$$\mathcal{J}(\xi) = (|\xi_1|^{r'-2} \xi_1, |\xi_2|^{r'-2} \xi_2) \quad .$$

Therefore

$$\mathcal{U}_\delta(\xi)_i = \delta^{r'+1} \frac{2}{(r'+1)\mu} (\sqrt{2}/\mu)^{r'-2} |\xi_i|^{r'-2} \xi_i + O(\delta^{r'+1+1/r}) \quad , \quad i = 1, 2$$

i.e.

$$\mathcal{U}_\delta(\xi) \approx C_\delta J(\xi) \quad .$$

We are now tempted to believe that the permeability function can be written in the form (86).

First of all, we can not hope to have  $K$  that depends only on the geometry of the porous medium, as in the linear case, because in two above examples it depends on the flow index  $r$ .

We know that this conjecture is true in case of unidirectional flow and, up to some order of approximation, for a network of narrow channels.

To verify whether (86) is true or not, we define the conjugated permeability function

$$G(\xi)_i = |\mathcal{U}(\xi)_i|^{r-2} \mathcal{U}(\xi)_i \quad , \quad i = 1, \dots, n \quad .$$

As  $r-2 < 0$  and  $\mathcal{U}(0) = 0$ ,  $G(0)$  is not correctly defined by the above formula. However, since  $\mathcal{U}$  satisfies (82), and (83) we have

$$m|\xi| \leq |G(\xi)| \leq M|\xi| \quad .$$

Therefore we define  $G$  in  $\xi = 0$  by  $G(0) = 0$  and we get the function  $G \in C(\mathbf{R})^n$ . Function  $G$  is obviously homogeneous

$$G(\lambda \xi) = \lambda G(\xi) \quad .$$

we refer to the section 6.4 where a numerical example shows that in case of bundle of perpendicular, not necessarily narrow, channels  $G$  is differentiable and linear, i.e. that (86) can be expected to hold in rectangular geometry. That can be explained by the fact that we have  $n$  unidirectional flows with, almost, no interaction between them. In fact, example 2 proves that the interaction between two flows exists, but it has some lower order.

As it was noticed in [13] the function  $\mathcal{U}$  is differentiable for any  $\xi \neq 0$ . We are, therefore, in situation very close to (86). Moreover

$$\frac{\partial \mathcal{U}(\xi)_i}{\partial \xi_j} = \int_{\mathcal{Y}} [w_\xi^j(y)]_i dy \quad , \quad (93)$$

where

$$-\mu \operatorname{div}_y \{ (r-2) |e_y(w_\xi)|^{r-4} [e(w_\xi) \cdot e(w_\xi^j)] e_y(w_\xi) + \quad (94)$$

$$+ |e_y(w_\xi)|^{r-2} e_y(w_\xi^j) \} + \nabla_y \pi_\xi^j = \mathbf{e}_j \quad \text{in } \mathcal{Y} \quad (95)$$

$$\operatorname{div}_y w_\xi^j = 0 \quad \text{in } \mathcal{Y}$$

$$(w_\xi^j, \pi_\xi^j) \text{ is } \mathcal{Y} - \text{periodic}$$

$$w_\xi^j = 0 \quad \text{on } S \quad . \quad (96)$$

Since this problem is linear it is easy to prove the following result:

**Proposition 2** For  $\xi \neq 0$  the problem (95)-(96) has a unique solution  $w_\xi^i \in V_\xi$ , where

$$V_\xi = \{ \phi \in W_\#^{1,r}(\mathcal{Y})^n ; |e(w_\xi)|^{\frac{r}{2}-1} e(\phi) \in L^2(\Omega) \quad , \quad \operatorname{div} \phi = 0 \}$$

and

$$W_\#^{1,r}(\Omega) = \{ \phi \in W^{1,r}(\mathcal{Y}) ; w_\xi^j \text{ is } \mathcal{Y} - \text{periodic} \quad , \quad w_\xi^j = 0 \text{ on } S \} \quad .$$

Furthermore  $w_\xi^j$  satisfies the a priori estimate

$$|e(w_\xi^j)|_{L^r(\mathcal{Y})} \leq C |e(w_\xi)|_{L^r(\mathcal{Y})}^{2-r} \quad . \quad (97)$$

**Proof.** We first notice that the definition of  $V_\xi$  has a sense since, due to the (21),  $e(w_\xi) \neq 0$  (a.e) in  $\mathcal{Y}$ . Now  $V_\xi$  is the Banach space equipped by the norm

$$|\phi|_{V_\xi} = |e(\phi)|_{L^r(\mathcal{Y})} + |e(w_\xi)|^{\frac{r}{2}-1} e(\phi)|_{L^2(\mathcal{Y})} .$$

The operator  $T : V_\xi \rightarrow V'_\xi$ , defined by

$$T \phi = -\mu \operatorname{div}_y \{ (r-2) |e_y(w_\xi)|^{r-4} [e(w_\xi) \cdot e(\phi)] e_y(w_\xi) + |e_y(w_\xi)|^{r-2} e_y(\phi) \}$$

is coercive since, using the Hölder's inequality, we get

$$\begin{aligned} \langle T \phi | \phi \rangle &\geq (r-1) \int_{\mathcal{Y}} |e(w_\xi)|^{r-2} |e(\phi)|^2 \\ &\geq \frac{r-1}{2} \{ |e(w_\xi)|^{\frac{r}{2}-1} e(\phi)|_{L^2(\Omega)}^2 + (|e(w_\xi)|_{L^r(\mathcal{Y})})^{(r-2)} |e(\phi)|_{L^r(\mathcal{Y})}^2 \} . \end{aligned}$$

Now  $T \in \mathcal{L}(V_\xi, V'_\xi)$  is linear and coercive. Therefore it is bijective. The estimate (97) follows from

$$\langle T w_\xi^j | w_\xi^j \rangle \geq (r-1) (|e(w_\xi)|_{L^r(\mathcal{Y})})^{(r-2)} |e(w_\xi^j)|_{L^r(\mathcal{Y})}^2 . \clubsuit$$

**Lemma 5** *The matrix  $\nabla_\xi \mathcal{U}(\xi)$ ,  $\xi \neq 0$  is symmetric.*

**Proof.** Multiplying (95) by  $w_\xi^j$  and integrating over  $\mathcal{Y}$  we obtain

$$\begin{aligned} \frac{\partial \mathcal{U}_j}{\partial \xi_i}(\xi) &= \int_{\mathcal{Y}} \{ (r-2) |e_y(w_\xi)|^{r-4} [e(w_\xi) \cdot e(w_\xi^i)] [e_y(w_\xi) \cdot e_y(w_\xi^j)] + \\ &+ |e_y(w_\xi)|^{r-2} e_y(w_\xi^j) \cdot e_y(w_\xi^i) \} = \frac{\partial \mathcal{U}_i}{\partial \xi_j}(\xi) . \clubsuit \end{aligned}$$

Now  $\mathcal{U}$  is differentiable, its derivative is defined by (95)-(96) in any point  $\xi \in \mathbf{R}^n$  except for  $\xi = 0$  and it is symmetric. Consequently  $G$  is differentiable for any  $\xi \neq 0$  with

$$\frac{\partial G_j}{\partial \xi_i}(\xi) = (r-1) |\mathcal{U}(\xi)_j|^{r'-2} \frac{\partial \mathcal{U}_j}{\partial \xi_i}(\xi) . \quad (98)$$

Due to the estimate

$$\frac{|\mathcal{U}(\xi) - \mathcal{U}(0)|}{|\xi|} \leq M |\xi|^{r'-2}$$

we see that  $\nabla_\xi \mathcal{U}(0) = 0$ . It is, therefore, not possible to linearize  $\mathcal{U}$  in vicinity of  $\xi = 0$ , as we did in the Carreau's case. Once again we have the problem with  $\nabla_\xi G$  for  $\xi = 0$ , since (98) becomes an undetermined expression of the form  $\frac{0}{0}$ . But, we obviously have

$$|\nabla_\xi G(\xi)| \leq C ,$$

so that  $G \in C^{0,1}(\mathbf{R}^n)$ .

Unfortunately, this is as far as we go with (86). The two numerical examples found in section 6.4 show that, in general,  $G$  is nonlinear. That, of course, means that it is not differentiable for  $\xi = 0$ , which can also be seen from those numerical computations.

## 6 Numerical Experiments

To illustrate the various theoretical results of this article, we performe in this section numerical computations. After a short description of the numerical schemes used to solve the linear and nonlinear Stokes problem, we describe the different pores geometries considered. The following section (6.3) is devoted to the numerical experiment in case of Careau's viscosity, and finally the last section (6.4) is concerning the power-law viscosity.



## 6.1 Numerical methods used

We have to compute the solution of the nonlinear Stokes system (21–24) and for the case of the Carreau law viscosity, the solution of linear Stokes problems. Two difficulties have to be solved: the incompressibility condition and the nonlinearity in case of nonlinear viscosity law. An Augmented Lagrangian algorithm is used to satisfy the incompressibility condition  $\text{div } u = 0$ . We use a finite element method to discretize the Stokes problems: a  $P_2^\circ$  is used for the velocity field, and a  $P_1$  discontinuous element is used for the pressure field. Those elements are known to give good results with Augmented Lagrangian algorithm. Finally, the nonlinearities caused by the viscosity law are solved by a Newton Raphson procedure. We refer to [8] for the description of the global algorithm. The porous cells are discretized by a mesh containing 1000 triangles.

## 6.2 The Porous media considered

We consider three models of porous media to make numerical experiments: Each of them are characterized by the solid inclusion in the periodic unitary cell. As shown in figure (1), we consider three geometries of the inclusion: a cylindrical inclusion, a square inclusion, and finally, an ellipsoidal inclusion which gives an anisotropic porous medium. In the following, we refer to **GEOM1** for the porous medium with circular inclusion, **GEOM2** for this one with square inclusion and finally, **GEOM3** for the last porous medium with ellipsoidal inclusion.

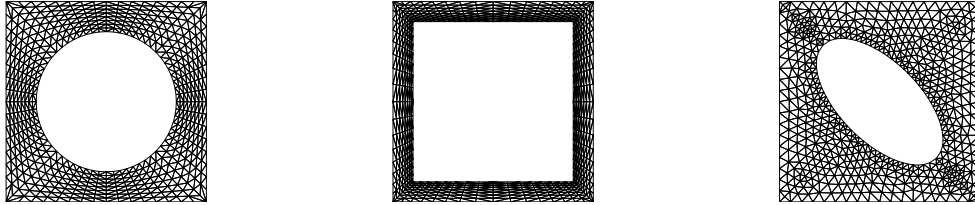


Figure 1: Periodic unitary cell: **GEOM1** on the left, **GEOM2** in the middle and finally **GEOM3** on the right of the figure.

## 6.3 Numerical Simulations for Carreau's viscosity law

In this section, we propose to solve (21–24) for a sampling of external force  $\xi$  and its Taylor's expansion around the origin when viscosity is governed by a Carreau's law. At first, we describe parameters of the Carreau law and the numerical experiments procedure. In applications (see e.g. [1])  $\eta_\infty$  is either very small compared to  $\eta_0$  (polymer solutions) or even equal to zero (polymer melts). For this reason, neglect  $\eta_\infty$  and consider a simplified Carreau law viscosity with  $\eta_\infty = 0$ . We chose arbitrarily  $\eta_0 = 1$  and consider two values of  $\lambda$ :  $\lambda_1 = 1$  and  $\lambda_2 = 100$  to take into account the different comportment for a viscosity law: one near a constant law, and the other one far of the constant case. Finally, we consider a flow index  $r = 1.5$ :

$$\eta(e(u)) = (1 + \lambda_i |e(u)|^2)^{-.25} \quad \text{with } \lambda_1 = 1. \text{ or } \lambda_2 = 100.$$

Because theoretical results are given for small  $\xi$ , we consider at first, the solution of the nonlinear problem (21–24) for  $|\xi| \leq 1$ . Numerical solution are performed for a sampling  $\xi_{ij}$  of  $\xi$  defined by:

$$\xi^{ij} = {}^t \left( \frac{i}{n} \cos \frac{j\pi}{4m}, \frac{i}{n} \sin \frac{j\pi}{4m} \right), 1 \leq i \leq n = 5, 0 \leq j \leq m = 8$$

extended by symmetry for  $m < j < 4m$ . For each value of  $\xi^{ij}$  we compute the gap  $\Delta_{ij}^1$  (resp.  $\Delta_{ij}^3, \Delta_{ij}^5$ ) between the solution  $\mathcal{U}_{ij}$  of (21–24) and the Taylor's expansion of order 1 (resp 3, 5) where  $\Delta_{ij}^k$  are defined

by:

$$\Delta_{ij}^1 = \left[ \sum_{k=1}^2 (\mathcal{U}_{ij} - K \xi^{ij})^2 \right]^{\frac{1}{2}} \left[ \sum_{k=1}^2 \mathcal{U}_k^2 \right]^{-\frac{1}{2}}, \quad (99)$$

$$\Delta_{ij}^3 = \left[ \sum_{k=1}^2 \left( \mathcal{U}_{ij} - K \xi^{ij} - \frac{1}{2} \lambda(r-2) \sum_{\ell,m,n,p=1}^2 H_{\ell m}^{np} \xi_{\ell}^{ij} \xi_m^{ij} \xi_n^{ij} \mathbf{e}_p \right)^2 \right]^{\frac{1}{2}} \left[ \sum_{k=1}^2 \mathcal{U}_k^2 \right]^{-\frac{1}{2}} \quad (100)$$

$$\begin{aligned} \Delta_{ij}^5 = & \left[ \sum_{k=1}^2 \left( \mathcal{U}_{ij} - K \xi^{ij} - \frac{1}{2} \lambda(r-2) \left\{ \sum_{\ell,m,n,p=1}^2 H_{\ell m}^{np} \xi_{\ell}^{ij} \xi_m^{ij} \xi_n^{ij} \mathbf{e}_p \right. \right. \right. \\ & \left. \left. + \sum_{\ell,m,n,p,q,r=1}^2 H_{\ell m n}^{pqr} \xi_{\ell}^{ij} \xi_m^{ij} \xi_n^{ij} \xi_p^{ij} \xi_q^{ij} \mathbf{e}_r \right\} \right)^2 \right]^{\frac{1}{2}} \left[ \sum_{k=1}^2 \mathcal{U}_k^2 \right]^{-\frac{1}{2}} \end{aligned} \quad (101)$$

On figure (2) we plot different  $\Delta_{ij}^k$  for  $k = 1, 2, 3$  for  $\lambda_2 = 100$  and for the first two geometries **GEOM1** and **GEOM2**. Concerning the first geometry, one may see that we need the third term in the Taylor's expansion to obtain good results. But, in the case of big square inclusion (**GEOM2**), for this range of  $|\xi|$ , the global filtration law is very close to the Darcy's law: there is no real gap between Darcy's law and the Taylor's expansion of any of three orders. To underline those comportments, we have drawn on

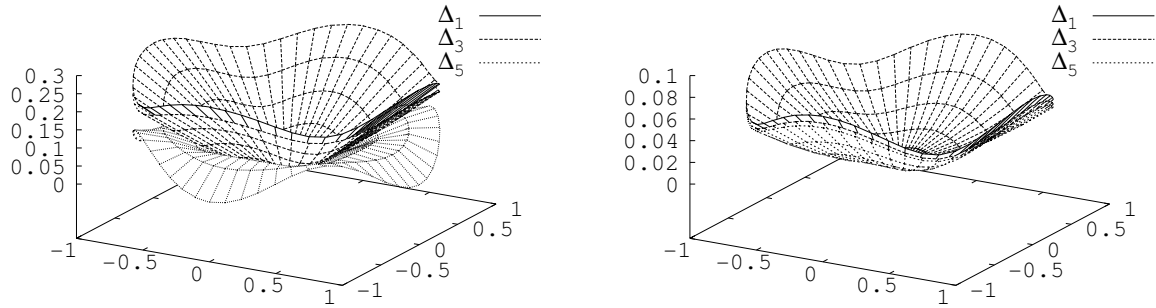


Figure 2: Gap between  $\mathcal{U}$  and its Taylor's expansions of order 1, 3 and 5 for **GEOM1** on the left, and for **GEOM2** on the right (for  $\lambda_2 = 100$ ).

figures (3-4) the same quantities (eq. (99–101)) for a fixed direction of  $\xi$  and for length  $0 \leq |\xi| \leq 1$ . Let consider first figure (3) concerning **GEOM1**. In the case of  $\lambda_1 = 1$  one may see that for the range of considered  $\xi$ , the filtration law is very close to Darcy's law: the relative errors, independently of order, are smaller than 1%. But in the case of  $\lambda_2 = 100$ , one needs the fifth approximation to obtain error smaller than 1%. Secondly, when considering figure (4) concerning **GEOM2**, one may see that independently of the value of  $\lambda$ , the behaviour of  $\mathcal{U}$  remains almost linear: for  $\lambda_1 = 1$  error is smaller than 0.8% and for  $\lambda_1 = 100$  error is smaller than 8%. To close this numerical study of theoretical results concerning Carreau's viscosity, we have computed solution of (21–24) and the different gaps (eq. (99–101)) for  $|\xi|$  larger than 1, to obtain numerical limit of validity of the Taylor's expansion, for the three geometries but only for one fixed direction of  $\xi$ :  $\angle(\mathbf{e}_1, \xi) = 60^\circ$ . Those results are plotted on figure 5. One may see that for the three geometries, Taylor's expansions give good results for  $\lambda_1 = 1$  for the considered range of  $\xi$ , but in the case of larger parameter  $\lambda_2 = 100$  one loses precision of the Taylor's expansion. More precisely, the best results are for the third order Taylor's expansion, but the fifth order expansion gives

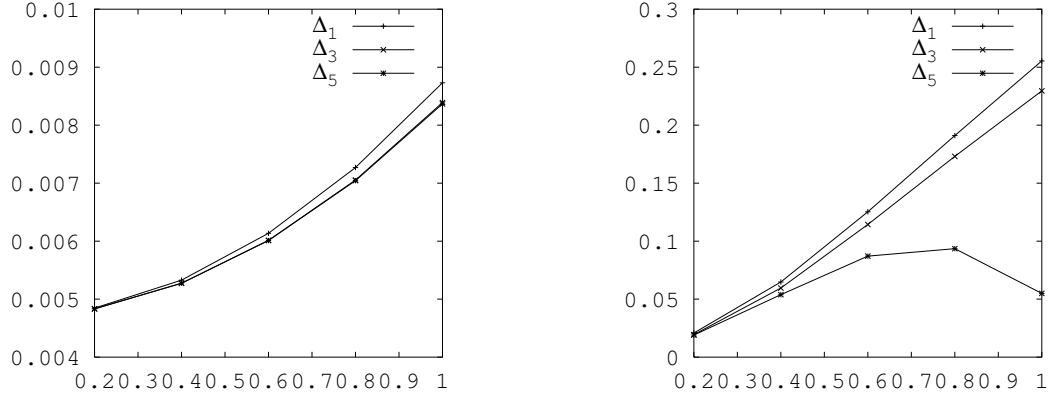


Figure 3: Gap between  $\mathcal{U}$  and its Taylor's expansion for **GEOM1**:  $|\xi| \leq 1$ ,  $(\mathbf{e}_1, \xi) = 45^\circ$ ,  $\lambda = 1$  on the left,  $\lambda = 100$  on the right.

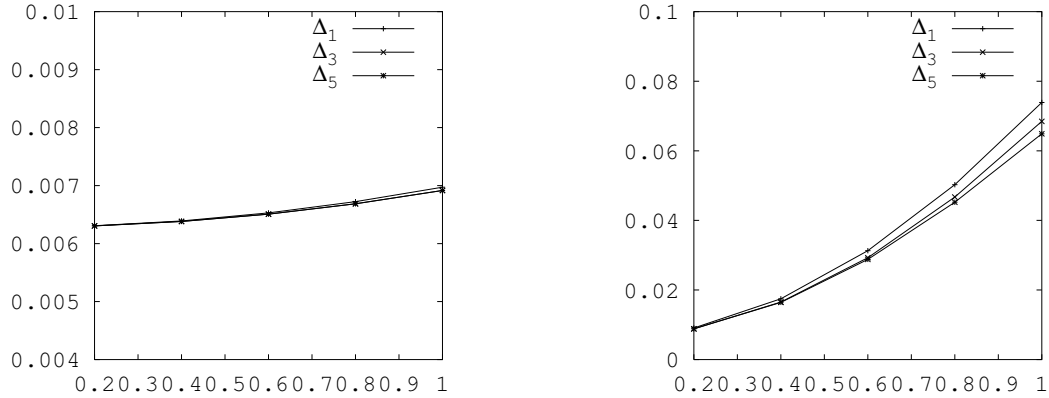


Figure 4: Gap between  $\mathcal{U}$  and its Taylor's expansion for **GEOM2**:  $|\xi| \leq 1$ ,  $(\mathbf{e}_1, \xi) = 45^\circ$ ,  $\lambda = 1$  on the left,  $\lambda = 100$  on the right.

very bad results ( the rest of the Taylor expansion of order 7 becomes non negligible). This suggests that for large  $|\xi|$  and  $\lambda$  the Taylor's series diverge.

## 6.4 Numerical simulation for Power viscosity law

In this section, we propose to solve (21–24) for a sampling of  $\xi$  when viscosity is governed by a Power law to illustrate the theoretical results of section 4. At first, we describe parameters of the Power law and the numerical experiments procedure. Due to the homogeneity property of the power law viscosity, we just consider a viscosity law with  $\lambda = 1$ . We chose arbitrarily a flow index  $r = 1.5$  :

$$\eta(e(u)) = |e(u)|^{-.5}$$

For the same reason of homogeneity, we consider only the solution of the nonlinear problem (21–24) for  $|\xi| = 1$ . Numerical solution are performed for a sampling  $\xi^j$  of  $\xi$  defined by:

$$\xi^j = {}^t(\cos \frac{j\pi}{4m}, \sin \frac{j\pi}{4m}), \quad 0 \leq j \leq m = 8 \quad (102)$$

and any extended by symmetry as in the Carreau's case. In two following sections, we will numerically show the different behaviour of the filtration law depending on the geometry of porous medium.

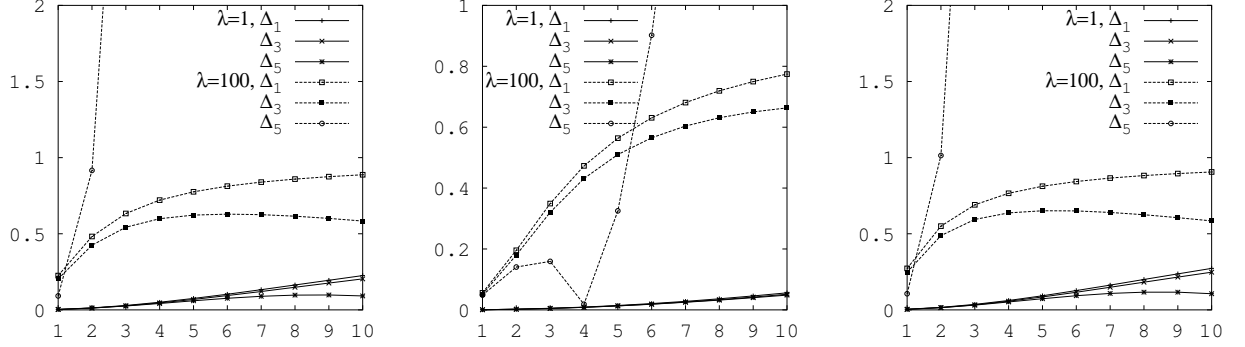


Figure 5: Gap between  $\mathcal{U}$  and its Taylor's development of order 1, 3 and 5 for  $|\xi| \leq 10$ , for **GEOM1** on the left, and for **GEOM2** in the middle and **GEOM3** on the right.

#### 6.4.1 Numerical results

This section is devoted to numerical results in the case of power law to exhibit the linear or nonlinear comportment of the function  $G(\xi)$  defined by its components  $G(\xi)_i = |\mathcal{U}_i(\xi)|^{r-2} \mathcal{U}_i(\xi)$ . On the figure (6) we plot the first component  $G_1$  of the function  $G$  for the three geometries  $\xi$  (due to the symmetry of inclusion, the comportment of the second component of  $G$  is deduced from the first one by an appropriate symmetry). It may be noticed that the comportment of  $G_i$  as a function of  $\xi_i$  seems linear for the case of the square inclusion (**GEOM2**), but one sees that in the case of the spherical or ellipsoidal inclusion (**GEOM1** and **GEOM3**), this is not the case. To underline this effect, we have computed the slope  $A$

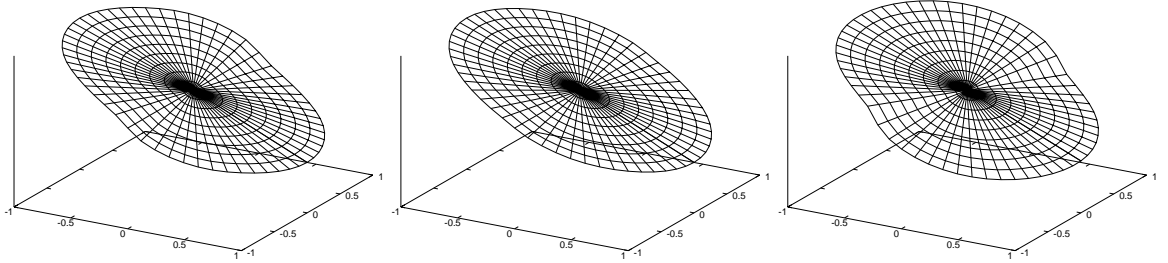


Figure 6: First component  $G_1(\xi)$  of the function  $G$  for **GEOM1** on left **GEOM2** in the middle and **GEOM3** on right

of the straight line  $y = A\xi_i$  which approaches, in the least square sense, the function  $G_i$  for the each of the three geometries. This slope is solution of the following problem:

$$\inf_{B \in \mathbf{R}} \sum_{j=1}^N [G_i(\xi^j) - B\xi_i^j]^2$$

and is given by:

$$A = \frac{\sum_{j=1}^N G_i(\xi^j) \xi_i^j}{\sum_{j=1}^N (\xi_i^j)^2}.$$

Then we have computed the relative distance  $\Delta$  between the straight line and the graph of  $G_i$ :

$$\Delta = \left[ \frac{\sum_{j=1}^N [G_i(\xi^j) - A\xi_i^j]^2}{\sum_{j=1}^N [G_i(\xi^j)]^2} \right]^{\frac{1}{2}}$$

In table1 we give the values of  $A$  and  $\Delta$ : One sees that  $\Delta$  is the smallest for the square inclusion

	$GEOM_1$	$GEOM_2$	$GEOM_3$
$A$	$-0.24083 \cdot 10^{-1}$	$-0.63016 \cdot 10^{-2}$	$-0.41984 \cdot 10^{-1}$
$\Delta$	$0.99781 \cdot 10^{-1}$	$0.44640 \cdot 10^{-1}$	$0.27976$

Table 1: Slope  $A$  of the straight line and relative distance  $\Delta$  between this line and  $G_1$  for the three geometries

(**GEOM2**), (of order 4%), and is no negligible for the other geometries (10% for the spherical inclusion and 27% for the ellipsoidal inclusion). Finally, we have drawn on figure (7), the pointwise distance between the straight line  $y = A\xi_1$  and the function  $G(\xi)_1 = |\mathcal{U}_1(\xi)|^{r-2}\mathcal{U}_1(\xi)$  for the three geometries. One may see that the gap between the straight line and the function  $G_1$  is maximal at the point where  $\mathcal{U}_1$  is changing sign (for angle  $\pi/2$  for **GEOM1** and **GEOM2**). This effect may be understood as the non-differentiability of  $\mathcal{U}$  in the origin.

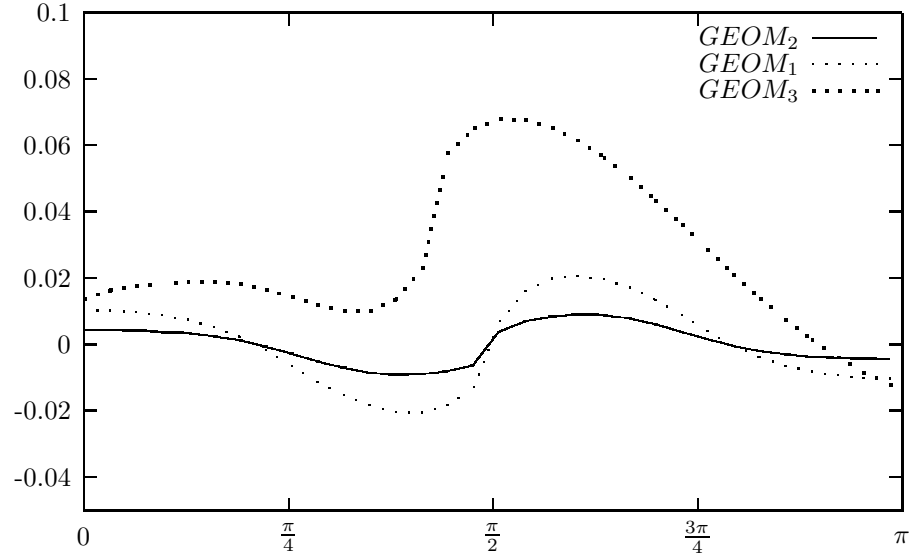


Figure 7: Pointwise gap between  $G_1$  and its least square approximation  $y = A\xi_1$  and  $G_1(\xi)$  for **GEOM1**, **GEOM2** and **GEOM3**.

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